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Stable intersections of affine cantor sets

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Abstract. We construct open sets of pairs of affine Cantor sets (K, K') with the simplest possible combinatorics (i.e., defined by two affine increasing maps) which have stable intersection, while the product of lateral thicknesses $\tau_R(K) \cdot \tau_L(K')$ is smaller than one. Thus, in a strong form, the converse to the following classic result due to Newhouse is not true: if the product of the thicknesses of two Cantor sets is bigger than one then there are translations of them which have stable intersection. We also describe the topological structure of $K - K'$ for typical pairs (K, K') in this open set.

Keywords: Affine Cantor sets, stable intersections, arithmetic differences, thickness, Palis conjecture.

Mathematical subject classification: 37C45, 28A80, 37G25.

1 Introduction

Regular Cantor sets play an important role in dynamical systems and in some problems in number theory, related to diophantine approximations. Intersections of hyperbolic sets with stable and unstable manifolds of its points are often regular Cantor sets, and so are many Cantor sets given by combinatorial conditions on the continued fraction of real numbers. The study of metrical and topological properties of arithmetic differences of regular Cantor sets appear naturally in the study of homoclinic bifurcations, in dynamical systems, and in the study of the classical Markov and Lagrange spectra related to diophantine approximations of real numbers, in number theory (see [PT] and [M2]). Motivated by the study of homoclinic bifurcations, J. Palis conjectured that generically the arithmetic difference of two regular Cantor sets either has zero measure or else contains an interval, and that this fact is always true for affine Cantor sets (see [PT], [M1] and [P1], [P2]). C. G. Moreira and J.-C. Yoccoz proved this conjecture in the setting of regular Cantor sets (see [MY]):

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Given two regular Cantor set whose sum of Hausdorff dimensions is greater than one, in almost all cases, there exist translations of them whose intersection persistently has positive Hausdorff dimension.

In fact they prove the existence of a non empty recurrent compact set of relative configurations of (most pairs of) these Cantor sets, a condition which implies stable intersection of pairs of Cantor sets in these relative configurations, as we explain in section 1. We recall that, if the sum of the Hausdorff dimensions of two regular Cantor sets is smaller than one, then their arithmetic difference has zero Lebesgue measure.

Is to be emphasized that the above conjecture remains open in the affine case, even in the generic setting. Besides the Hausdorff dimension, there is another fractal invariant, introduced by Newhouse in [N], that plays a relevant role in such questions, namely the thickness (which we define below). Newhouse showed that the arithmetic difference of two Cantor sets whose product of thicknesses is larger than one contains nontrivial intervals. Such thickness condition was generalized in [M1]. In this paper we provide open sets of pairs of affine Cantor sets with the simplest possible combinatorics (defined by two affine expansive maps) which have stable intersection in many relative positions (and so their arithmetic differences persistently contains nontrivial intervals), but do not satisfy the generalized thickness conditions of [M1]. We also study the topological structure of the arithmetic difference of such sets, which is generically an R-Cantorval, a kind of compact set with dense interior and fractal boundary, in contrast with the examples given by the thickness conditions, where arithmetic differences are finite unions of intervals.

Let us give some initial definitions in order to state more precisely our results. We say that a Cantor set K is regular or dynamically defined if:

i) there are disjoint compact intervals $K_1, K_2, \cdots, K_r$ such that $K \subseteq K_1 \cup \cdots \cup K_r$ and the boundary of each K_i is contained in K,

ii) there is a $C^{1+\epsilon}$ expanding map ψ defined in a neighborhood of $K_1 \cup K_2 \cup \cdots \cup K_r$ such that $\psi(K_i)$ is the convex hull of a finite union of some intervals K_j satisfying:

ii.1 For each i, $1 \leq i \leq r$ and n sufficiently big, $\psi^n(K \cap K_i) = K$,

ii.2 $K = \cap_{n=1}^{\infty} \psi^{-n}(K_1 \cup K_2 \cup \cdots \cup K_r)$.

The set $\{K_1, K_2, \cdots, K_r\}$ is, by definition, a Markov partition for K, and the set $D = \cup_{i=1}^{r} K_i$ is the corresponding Markov domain of K.

We say that the Cantor set K is close on the topology $C^{1+\epsilon}$ to a Cantor set $\widetilde{K}$ with Markov partition $\{\widetilde{K_1}, \widetilde{K_2}, \cdots, \widetilde{K_s}\}$ defined by the expansive map $\widetilde{\psi}$ if and only if $r = s$, the extremes of K_i are near the corresponding extremes of $\widetilde{K_i}$, $i = 1, 2, ..., r$ and supposing $\psi \in C^{1+\epsilon}$ with Holder constant C, we must have $\widetilde{\psi} \in C^{1+\widetilde{\epsilon}}$ with Holder constant $\widetilde{C}$ such that $(\widetilde{C}, \widetilde{\epsilon})$ is near (C, ϵ) and $\widetilde{\psi}$ is close to ψ in the C^1 topology.

We say that regular Cantor sets K and K' have stable intersection if for any pair of regular Cantor sets $(\widetilde{K}, \widetilde{K'})$ near (K, K'), we have $\widetilde{K} \cap \widetilde{K'} \neq \emptyset$.

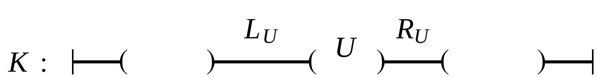
Stable intersections between Cantor sets which come from stable and unstable foliations of a horseshoe provide persistent examples of open sets of non-hyperbolic diffeomorphisms, after the unfolding of a homoclinic tangency. In these cases, the open set of diffeomorphisms presenting persistent tangencies between the stable and unstable foliations of the horseshoe stably has positive lower density at the initial parameter of the bifurcation, in parametrized families (see [M1] and [PT]).

Stable intersection of regular Cantor sets K and K' clearly implies the existence of an interval contained in the arithmetic difference of the Cantor sets K and K', which is given by:

$$K - K' := \{x - y \mid x \in K, y \in K'\} = \{t \in \mathbb{R} \mid K \cap (K' + t) \neq \emptyset\}.$$

The lateral thicknesses τ_R and τ_L and the thickness τ of a Cantor set are defined as follows:

Let U be a gap (i.e., a connected component of the complement) of the Cantor set K and let L_U , R_U be the intervals at its left and its right, respectively, that separate it from the closest larger gaps.



We define $\tau_R(U) = \frac{|R_U|}{|U|}$, $\tau_L(U) = \frac{|L_U|}{|U|}$,

$\tau_R(K) =$Inf$\{\tau_R(U)|U$ bounded gap of $K\}$, the right thickness of K,

$\tau_L(K) =$Inf$\{\tau_L(U)|U$ bounded gap of $K\}$, the left thickness of K and

$\tau(K) =$Min$\{\tau_R(K), \tau_L(K)\}$, the thickness of K (introduced by Newhouse).

In [M1], it is shown that there exist an open and dense subset in $C^{1+\epsilon}$ topology of regular Cantor sets whose elements have these lateral thicknesses varying continuously. It is also shown that these lateral thicknesses are continuous at affine cantor sets defined by two expansive maps.

Here we state some classical results on regular Cantor sets:

I) If $HD(K) + HD(K') < 1$, then $K - K'$ has measure zero, therefore no translations of K and K' have stable intersection (see [PT]).

II) If $\tau(K).\tau(K') > 1$ and K is linked to K' then (K, K') have stable intersection, therefore $K - K'$ contains a nontrivial interval (see [N] and [PT]).

III) If $\tau_R(K).\tau_L(K') > 1$ and $\tau_L(K).\tau_R(K') > 1$ then $K - K'$ always contains an interval. In fact (K, K') have stable intersection, provided

i) K is linked to K',

ii)τ_R and τ_L are continuous at K and K' (see [M1]).

IV) There exist Cantor sets K, K' such that $K - K'$ has positive Lebesgue measure, but doesn't contain any interval (see [S]).

It is not difficult to construct affine Cantor sets defined by more than 2 expansive maps and small lateral thicknesses, while they have stable intersection (see [M1]). However, for affine Cantor sets defined by two affine expansive map, the conditions given in [M1] which guarantee stable intersections are equivalent to condition (III) above. These considerations motivate the following problem, which we will discuss in this work:

Problem 1. *Does there exist a non empty open set in the space of affine Cantor sets defined by two expansive maps contained in the region*

$$\{(K, K') \mid \tau_R(K).\tau_L(K') < 1 \textit{ or } \tau_L(K).\tau_R(K') < 1\},$$

such that their elements have stable intersection?

In order to give an affirmative solution to this problem, the main challenge is to construct a recurrent compact set of relative configurations. It would give a positive answer to Problem 1, since, by the proposition in section 2.3 of [MY], any relative configuration contained in a recurrent compact set is a configuration of stable intersection.

In [MMR], the topological structures with appear persistently for arithmetic sums and differences of regular Cantor sets are studied. We apply some techniques of this work in order to show that for generic pairs (K, K') of Cantor sets in the open set which we are considering, the arithmetic difference $K - K'$ is an R-Cantorval (a certain type of compact subset of the real line introduced in [MO], a work about the topological structure of arithmetic sums of affine Cantor sets).

In section 2 we present the necessary definitions and state the recurrence condition on relative configurations of [MY] which implies stable intersection of pairs of regular Cantor sets.

In section 3 we translate this condition to the setting of affine Cantor sets, which gives a recurrent condition on a simpler space of relative configurations.

In section 4 we construct a recurrent set of relative configurations of affine Cantor sets defined by two expansive maps K and K' with $\tau_L(K) \cdot \tau_R(K') < 1$ under special conditions (theorem 2).

In section 5 we characterize the topological structure of arithmetic difference sets $K - \lambda K'$ with $\lambda > 0$ for generic pairs (K, K') as described in section 4.

2 Basic definitions

We will use notations similar to those of [MY] which are restated here.

A regular Cantor set can be reinterpreted as follow.

Let A be a finite alphabet, $\mathcal{B}$ a subset of A^2, and Σ the subshift of finite type $A^{\mathbb{Z}}$ with allowed transitions $\mathcal{B}$.

We will always assume that Σ is topologically mixing, and that every letter in A occurs in Σ.

An *expansive map of type* Σ is a map g with the following properties:

i) the domain of g is a disjoint union $\bigcup_{\mathcal{B}} I(a, b)$, where, for each (a, b), $I(a, b)$ is a compact subinterval of $I(a) := [0, 1] \times \{a\}$,

ii) for each $(a, b) \in \mathcal{B}$, the restriction of g to $I(a, b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all t.

The *regular Cantor set* associated to g is the maximal invariant set

$$K = \bigcap_{n \geq 0} g^{-n}\left(\bigcup_{\mathcal{B}} I(a, b)\right).$$

A regular Cantor set K is *affine* if Dg is constant on every $I(a, b)$.

Let $\Sigma^- = \{(\theta_n)_{n \leq 0} : (\theta_i, \theta_{i+1}) \in \mathcal{B} \textit{ for } i < 0\}$. We equip Σ^- with the following ultrametric distance: for $\underline{\theta} \neq \underline{\tilde{\theta}} \in \Sigma^-$, set

$$d(\underline{\theta}, \underline{\tilde{\theta}}) = \begin{cases} 1 & \theta_0 \neq \tilde{\theta}_0 \\ |I(\underline{\theta} \wedge \underline{\tilde{\theta}})| & \textit{otherwise} \end{cases},$$

where $\underline{\theta} \wedge \underline{\tilde{\theta}} = (\theta_{-n}, \cdots, \theta_0)$ if $\tilde{\theta}_{-j} = \theta_{-j}$ for $0 \leq j \leq n$ and $\tilde{\theta}_{-n-1} \neq \theta_{-n-1}$. Now, let $\underline{\theta} \in \Sigma^-$; for $n > 0$, let $\underline{\theta}^n = (\theta_{-n}, \ldots, \theta_0)$, and let $B(\underline{\theta}^n)$ be the affine map from $I(\underline{\theta}^n)$ onto $I(\theta_0)$ such that the diffeomorphism $k_n^{\underline{\theta}} = B(\underline{\theta}^n) \circ f_{\underline{\theta}^n}$ is orientation preserving. For any $\underline{\theta} \in \Sigma^-$, there is a smooth diffeomorphism $k^{\underline{\theta}}$ such that $k_n^{\underline{\theta}}$ converges to $k^{\underline{\theta}}$ in $\mathrm{Diff}_+^r(I(\theta_0))$, for any $r \in (1, +\infty)$, uniformly in $\underline{\theta}$.

Next, we define renormalization operators. For $(a, b) \in \mathcal{B}$, let

$$f_{a,b} = [g|_{I(a,b)}]^{-1};$$

this is a contracting diffeomorphism from $I(b)$ onto $I(a, b)$. If $\underline{a} = (a_0, a_1, \ldots, a_n)$ is a word of Σ, we put

$$f_{\underline{a}} = f_{a_0,a_1} \circ \cdots f_{a_{n-1},a_n},$$

this is a contracting diffeomorphism from $I(a_n)$ onto a subinterval of $I(a_0)$ that we denote by $I(\underline{a})$.

Let $F^{\underline{\theta}}$ be the affine map from $I(\theta_0)$ onto $I(\theta_{-1}, \theta_0)$ with the same orientation of f_{θ_{-1},θ_0}.

Let $\mathcal{A} = \{(\underline{\theta}, A)\}$, where $\underline{\theta} \in \Sigma^-$ and A is an affine embedding of $I(\theta_0)$ into $\mathbb{R}$. The *renormalization operators* $T_{\theta_1,\theta_0} : \mathcal{A} \to \mathcal{A}$ is defined as follow:

$$T_{\theta_1,\theta_0}(\underline{\theta}, A) = (\underline{\theta}\theta_1, A \circ F^{\underline{\theta}\theta_1}) \text{ where } (\theta_0, \theta_1) \in \mathcal{B}.$$

Assume that we are given two sets of data $(A, \mathcal{B}, \Sigma, g)$, $(A', \mathcal{B}', \Sigma', g')$ defining regular Cantor sets K, K'.

We denote by C the quotient of $\mathcal{A} \times \mathcal{A}$ by the diagonal action on the left of affine group.

A compact set $\mathcal{L}$ in C is *recurrent* if for every $u \in \mathcal{L}$ and $\ell, \ell' \geq 0$ with $\ell + \ell' > 0$, when $(\underline{\theta}, A)$ and $(\underline{\theta}', A')$ represents u, then there exists words $\underline{a} = (a_0, \cdots, a_\ell)$ and $\underline{a'} = (a'_0, \cdots, a'_{\ell'})$ in Σ and Σ', respectively, with $a_0 = \theta_0$, $a'_0 = \theta'_0$, such that $(T_{\underline{a}}(\underline{\theta}, A), T'_{\underline{a'}}(\underline{\theta'}, A')) = v$ belongs to $\mathcal{L}^\circ := int(\mathcal{L})$. The following proposition is proved in [MY].

Proposition 1. *Any relative configuration (of limit geometries) contained in a recurrent compact set is stably intersecting.*

In the end of this section we suppose that $S = \Sigma^- \times \Sigma'^- \times \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. One can see that the fibers of the quotient map $C \longrightarrow S$ are one-dimensional and have a canonical affine structure. Moreover, this bundle map is trivializable. We choose an explicit trivialization $C \cong S \times \mathbb{R}$ in order to have a coordinate in each fiber.

3 Transfer of renormalization operators on the space $S \times \mathbb{R}$

We assume that K is an affine Cantor set together with the Markov partition $\{I(n,m)\}_{m,n\in A}$ and an expansive map

$$\Phi\mid_{I(n,m)}(x) = p_{(n,m)} \cdot x + q_{(n,m)},$$

where $A = \{1,2,3,\ldots,N\}$. Moreover, we assume that $I(n) = [a_n^1, a_n^2]$, $I(n,m) = [a_{n,m}^1, a_{n,m}^2]$, therefore, for every $(n,m) \in \mathcal{B}$, we have:

$$\Phi(I(n,m)) = I(m).$$

In the case $p_{(\theta_0,\theta_1)} > 0$, $F^{\underline{\theta}\theta_1}$ is as below:

$$F^{\underline{\theta}\theta_1} : I(\theta_1) \longrightarrow I(\theta_0,\theta_1),$$

$$F^{\underline{\theta}\theta_1}(x) = \frac{a^2_{\theta_0,\theta_1} - a^1_{\theta_0,\theta_1}}{a^2_{\theta_1} - a^1_{\theta_1}}(x - a^1_{\theta_1}) + a^1_{\theta_0,\theta_1}.$$

Also, in the opposite orientation, we have:

$$F^{\underline{\theta}\theta_1}(x) = \frac{a^2_{\theta_0,\theta_1} - a^1_{\theta_0,\theta_1}}{a^2_{\theta_1} - a^1_{\theta_1}}(x - a^1_{\theta_1}) + a^2_{\theta_0,\theta_1}.$$

Therefore, in both cases we have:

$$F^{\underline{\theta}\theta_1}(x) = \frac{1}{p_{(\theta_0,\theta_1)}}x - \frac{q_{(\theta_0,\theta_1)}}{p_{(\theta_0,\theta_1)}}.$$

In this section we first construct the homeomorphism between $S \times \mathbb{R}$ and C, then we transfer all renormalization operators to $S \times \mathbb{R}$ and we find the recurrent set in $S \times \mathbb{R}$.

Theorem 1. *The map*

$$L : C \to S \times \mathbb{R}$$
$$\left[(\underline{\theta}, ax+b), (\underline{\theta}', a'x+b')\right] \mapsto \left(\underline{\theta}, \underline{\theta}', \frac{a'}{a}, \frac{b'-b}{a}\right)$$

is an homeomorphism between the space of relative configurations C and $S \times \mathbb{R}$.

Proof. L is well defined:
Let

$$[(\underline{\theta}, ax+b), (\underline{\theta}', a'x+b')] = [(\underline{\theta}, a_1x+b_1), (\underline{\theta}', a_1'x+b_1')].$$

Then there exists $c, d \in \mathbb{R}$ such that:

$$a_1x+b_1 = c(ax+b)+d \ , \ a_1'x+b_1' = c(a'x+b')+d,$$

therefore

$$\frac{a'}{a} = \frac{a_1'}{a_1} \ , \ \frac{b'-b}{a} = \frac{b_1'-b_1}{a_1}.$$

L is onto:

$$\forall(\underline{\theta}, \underline{\theta}', s, t) \in S \times \mathbb{R}, \qquad L([(\underline{\theta}, x), (\underline{\theta}', sx+t)]) = (\underline{\theta}, \underline{\theta}', s, t).$$

L is one to one:
Suppose that

$$L([(\underline{\theta}, x), (\underline{\theta}', sx+t)]) = L([(\underline{\theta}, x), (\underline{\theta}', s'x+t')]),$$

then $s = s'$, $t = t'$.
From the structure of L, we have L and L^{-1} are continuous. $\square$

Now we transfer the renormalization operators of relative configurations C to the space $S \times \mathbb{R}$. To do this, let $((\theta_1, \theta_0), (\theta_1', \theta_0')) \in \mathcal{B} \times \mathcal{B}'$, then we have:

$$(\underline{\theta}, \underline{\theta}', s, t) \xrightarrow{L^{-1}} \big[(\underline{\theta}, x) \ , \ (\underline{\theta}', sx+t)\big] \xrightarrow{(T_{\theta_1,\theta_0}, T'_{\theta_1',\theta_0'})}$$

$$\left[(\underline{\theta}\ \theta_1, \frac{1}{p_{(\theta_0,\theta_1)}}x - \frac{q_{(\theta_0,\theta_1)}}{p_{(\theta_0,\theta_1)}}) \ , \ (\underline{\theta}'\ \theta_1', \frac{s}{p_{(\theta_0',\theta_1')}}x - \frac{q_{(\theta_0',\theta_1')}}{p_{(\theta_0',\theta_1')}}s + t)\right] \xrightarrow{L}$$

$$\left(\underline{\theta}\ \theta_1, \underline{\theta}'\ \theta_1', \frac{p_{(\theta_0,\theta_1)}}{p'_{(\theta_0',\theta_1')}}\ s \ , \ p_{(\theta_0,\theta_1)}t - \frac{q'_{(\theta_0',\theta_1')}}{p'_{(\theta_0',\theta_1')}} p_{(\theta_0,\theta_1)}\ s + q_{(\theta_0,\theta_1)}\right),$$

we denote $L \circ (T_{\theta_1,\theta_0}, T'_{\theta_1',\theta_0'}) \circ L^{-1}$ by $T_{((\theta_1,\theta_0),(\theta_1',\theta_0'))}$. Moreover,

$$T_{((\theta_1,\theta_0),id)} : (\underline{\theta}, \underline{\theta}', s, t) \longrightarrow (\underline{\theta}\theta_1, \ \underline{\theta}', \ p_{\theta_1,\theta_0} s \ , \ p_{\theta_1,\theta_0}\ t + q_{\theta_1,\theta_0}),$$

$$T'_{(id,(\theta_1',\theta_0'))} : (\underline{\theta}, \underline{\theta}', s, t) \longrightarrow (\underline{\theta}, \ \underline{\theta}'\theta_1', \ \frac{s}{p'_{\theta_1',\theta_0'}} \ , \ t - \frac{q'_{\theta_1,\theta_0}}{p'_{\theta_1,\theta_0}} s).$$

4 Construction of recurrent sets

In this section we introduce a family of affine Cantor sets that have a recurrent compact set in relative configurations while they satisfy: $\tau_L(K) \cdot \tau_R(K') = \frac{3}{4} < 1$.

Theorem 2. *Suppose that K, K' are two Cantor sets as follows:*

That is, K is the affine Cantor set associated to

$$\psi : [0,1] \cup [3, n+3] \to [0, n+3]$$

given by

$$\psi(x) := \begin{cases} (n+3)x & x \in [0,1] \\ (n+3)(x-3)/n & x \in [3, n+3] \end{cases},$$

while K' is the affine Cantor set associated to $\varphi : [0,3] \cup [5,8] \to [0,8]$ given by

$$\varphi(x) := \begin{cases} 8x/3 & x \in [0,3] \\ 8(x-5)/3 & x \in [5,8] \end{cases}.$$

Then for $n \geq 130$, there exist a non empty compact recurrent set in relative configurations.

Proof. By Theorem 1, it is sufficient to show the existence of a recurrent compact set for the following maps:

$$\mathbb{R}^* \times R \to R^* \times R$$

$$(s,t) \xrightarrow{T_0} ((n+3)s, (n+3)t) \qquad (s,t) \xrightarrow{T_1} \left(\frac{n+3}{n}s, \frac{n+3}{n}t - \frac{3(n+3)}{n}\right) \quad (*)$$

$$(s,t) \xrightarrow{T_0'} \left(\frac{3}{8}s, t\right) \qquad (s,t) \xrightarrow{T_1'} \left(\frac{3}{8}s, t+5s\right)$$

We claim that if n is chosen conveniently large, then the set

$$\mathcal{L} = \{(s,t) | \ 1 \leq s \leq \frac{3(n+3)}{8}\ , \ 8s + t \geq 7 \ \text{and} \ t \leq n+2\},$$

is a recurrent set for the maps $(*)$.

Case 1. If $1 \leq s \leq \frac{n+3}{3}$ and $3 \leq t \leq n+2$, then $T_1(s,t) \in \mathcal{L}^\circ$, where $n \geq 24$.

Case 2. If $\frac{n+3}{3} < s \leq \frac{3(n+3)}{8}$ and $3 \leq t \leq n+2$, then $T_1 \circ T_0'(s,t) \in \mathcal{L}^\circ$.

Case 3. If $3 \leq s \leq \frac{3(n+3)}{8}$, $t < 3$ and $3s + t \geq 8$, then $T_0'(s,t) \in \mathcal{L}^\circ$.

Case 4. For $1 \leq s \leq 3$, $t < 3$ and $8s + t \geq 10$, we have:

$$8(\pi_1 \circ T_1(s,t)) + \pi_2 \circ T_1(s,t) = \frac{n+3}{n}(8s + t - 3) \geq 7(\frac{n+3}{n}) > 7.$$

Thus $T_1(s,t) \in \mathcal{L}^\circ$.

Case 5. If $1 \leq s < 3$, $7 \leq 8s + t < 10$, we denote:

$$i(s) := \begin{cases} 2 & 1 \leq s \leq \frac{256}{135} \\ 3 & \frac{256}{135} < s < 3 \end{cases}, \; c(s) := \frac{3^{i(s)}}{8^{i(s)-1}} s \quad \text{and} \quad m := \frac{n}{n+3}.$$

By using $\frac{4}{5} < c(s) \leq \frac{32}{15}$, we select n sufficiently large (e.g. $n \geq 130$) such that:

1) $10 - 3 - 3m - 3m^2 - c(s) < \frac{n^3(n+2)}{(n+3)^4}$ 2) $4 - 3m > \frac{8n}{(n+3)^2}$

3) $3m^2 + \frac{8}{n+3}m^3 - \frac{8}{3}c(s) < \frac{n^2(n+2)}{(n+3)^3}$ 4) $4 - \frac{5}{3}c(s) > \frac{8n}{(n+3)^2}$

5) $\frac{27}{32} < \frac{n-7}{n+3}m^2$ 6) $3 + 3m - \frac{17n-143}{3(n+3)}m^2 < \frac{n+1}{n+3}$

7) $3 + \frac{n+1}{n+3}m - \frac{2}{3}(\frac{2n+16}{n+3} + 3) > \frac{7}{n+3}$.

Now we consider following operators:

I) $T_1'^{i(s)} \circ T_0 \circ T_1^3(s,t) = (\frac{(n+3)^4}{n^3}(\frac{3}{8})^{i(s)}s, \frac{(n+3)^4}{n^3}(8s+t-3-3m-3m^2-c(s)))$,

II) $T_1'^{i(s)} \circ T_0 \circ T_1^2(s,t) = (\frac{(n+3)^3}{n^2}(\frac{3}{8})^{i(s)}s, \frac{(n+3)^3}{n^2}(8s + t - 3 - 3m - c(s)))$,

III) $T_0' \circ T_1'^{i(s)-1} \circ T_0 \circ T_1^2(s,t) = (\frac{(n+3)^3}{n^2}(\frac{3}{8})^{i(s)}s, \frac{(n+3)^3}{n^2}(8s+t-3-3m-\frac{8}{3}c(s)))$,

IV) $T_0' \circ T_1'^{i(s)-1} \circ T_0 \circ T_1(s,t) = (\frac{(n+3)^2}{n}(\frac{3}{8})^{i(s)}s, \frac{(n+3)^2}{n}(8s + t - 3 - \frac{8}{3}c(s)))$.

The above equalities hold, since for every natural number j we have:

i) $T_1^j(s,t) = ((\frac{n+3}{n})^j s, (\frac{n+3}{n})^j (t - 3 - 3m - 3m^2 - ... - 3m^{j-1}))$,

ii) $T_1^{'j}(s,t) = ((\frac{3}{8})^j s, t + 8(1 - (\frac{3}{8})^j)s)$.

For simplicity we use the notion $(\widetilde{s},\widetilde{t})$ as the image of operators (I),...,(IV). It is easy to see that for $n \geq 8$, we always have:

$$1 < \widetilde{s} < \frac{3(n+3)}{8} \quad \text{since} \quad \frac{4}{5} < c(s) \leq \frac{32}{15}.$$

To prove that $\mathcal{L}$ is recurrent, we have to divide this region to the following cases:

5.1) $8s + t \geq 3 + 3m + 3m^2 + \frac{8}{n+3}m^3$.

We use (I) and (1) and we obtain $T_1^{'i(s)} \circ T_0 \circ T_1^3(s,t) \in \mathcal{L}^\circ$, since

$$8\widetilde{s} + \widetilde{t} = \frac{(n+3)^4}{n^3}(8s + t - 3 - 3m - 3m^2) \geq \frac{(n+3)^4}{n^3}\frac{8}{n+3}(\frac{n}{n+3})^3 > 7.$$

5.2) $8s + t \leq 3 + 3m + \frac{n+1}{n+3}m^2 + c(s)$.

We use (II) and (2) and we obtain $T_1^{'i(s)} \circ T_0 \circ T_1^2(s,t) \in \mathcal{L}^\circ$, since

$$8\widetilde{s} + \widetilde{t} = \frac{(n+3)^3}{n^2}(8s + t - 3 - 3m).$$

5.3) $8s + t \geq 3 + 3m + \frac{8}{n+3}m^2 + \frac{5}{3}c(s)$ and we are not in the case (5.1).

We use (III) and (3) and we obtain $T_0' \circ T_1^{'i(s)-1} \circ T_0 \circ T_1^2(s,t) \in \mathcal{L}^\circ$, since

$$\begin{aligned}
8\widetilde{s} + \widetilde{t} &= \frac{(n+3)^3}{n^2}\left(8s + t - 3 - 3m - \frac{5}{3}c(s)\right) \\
&\geq \frac{(n+3)^3}{n^2}\frac{8}{n+3}\left(\frac{n}{n+3}\right)^2 > 7, \\
\widetilde{t} &= \frac{(n+3)^3}{n^2}\left(8s + t - 3 - 3m - \frac{8}{3}c(s)\right) \\
&< \frac{(n+3)^3}{n^2}(3m^2 + \frac{8}{n+3}m^3 - \frac{8}{3}c(s)) < n + 2.
\end{aligned}$$

5.4) $8s + t \leq 3 + \frac{n+1}{n+3}m + \frac{8}{3}c(s)$.

We use (IV) and (4) to obtain $T_0^{'} \circ T_1^{'i(s)-1} \circ T_0 \circ T_1(s,t) \in \mathcal{L}^{\circ}$, since

$$8\widetilde{s} + \widetilde{t} = \frac{(n+3)^2}{n}(8s + t - 3 - \frac{5}{3}c(s)) \geq \frac{(n+3)^2}{n}(4 - \frac{5}{3}c(s)) > 7,$$

$$\widetilde{t} = \frac{(n+3)^2}{n}(8s + t - 3 - \frac{8}{3}c(s)) \leq \frac{(n+3)^2}{n}\frac{n+1}{n+3}\frac{n}{n+3} < n+2.$$

If $i(s) = 3$, because (5) is valid we have:

$$3 + 3m + \frac{8}{n+3}m^2 + \frac{5}{3}c(s) < 3 + 3m + \frac{n+1}{n+3}m^2 + c(s).$$

Otherwise:

5.5) $Max\left\{3 + 3m + \frac{n+1}{n+3}m^2 + c(s),\ 3 + \frac{n+1}{n+3}m + \frac{8}{3}c(s)\right\} < 8s + t$

$$< Min\left\{3 + 3m + \frac{8}{n+3}m^2 + \frac{5}{3}c(s),\ 3 + 3m + 3m^2 + \frac{8}{n+3}m^3\right\}.$$

As

$$3 + 3m + \frac{n+1}{n+3}m^2 + c(s) < 8s + t < 3 + 3m + \frac{8}{n+3}m^2 + \frac{5}{3}c(s),$$

we obtain:

$$\frac{3(n-7)}{2(n+3)}m^2 < c(s).$$

Also from relation

$$3 + \frac{n+1}{n+3}m + \frac{8}{3}c(s) < 8s + t < 3 + 3m + 3m^2 + \frac{8}{n+3}m^3,$$

we have:

$$c(s) < \frac{2n+8}{n+3}m + 3m^2 + \frac{8}{n+3}m^3 < \frac{3}{8}\left(\frac{2n+16}{n+3} + 3\right).$$

Therefore we have:

$$T_1^{'} \circ T_0^{'} \circ T_0(s,t) = ((n+3)(\frac{3}{8})^2 s,\ (n+3)(8s + t - \frac{49}{9}c(s))) \in \mathcal{L}^{\circ},$$

since relations (i) and (ii) are valid:

i) $8s + t - \frac{49}{9}c(s) = 8s + t - \frac{5}{3}c(s) - \frac{34}{9}c(s)$

$$< 3 + 3m + \frac{8}{n+3}m^2 - \frac{17}{3}\frac{n-7}{n+3}m^2$$

$$= 3 + 3m - \frac{17n - 143}{3(n+3)}m^2$$

$$< \frac{n+1}{n+3},$$

since (6) is valid.

ii) $8(\pi_1 \circ T_1^{'} \circ T_0^{'} \circ T_0(s,t)) + \pi_2 \circ T_1^{'} \circ T_0^{'} \circ T_0(s,t) = (n+3)(8s+t-\frac{40}{9}c(s))$ $= (n+3)(8s+t-\frac{8}{3}c(s)-\frac{16}{9}c(s)) > (n+3)(3+\frac{n+1}{n+3}m-\frac{2}{3}(\frac{2n+16}{n+3}+3)) > 7,$ since (7) is valid.

Case 6. For $3 \le s \le \frac{3(n+3)}{8}$, $7 \le 8s+ t$ and $3s + t < 8$, we have:

i) $1 < \pi_1 \circ T_1^{'}(s,t) = \frac{3}{8}s < \frac{3(n+3)}{8}$,

ii) $\pi_2 \circ T_1^{'}(s,t) = t + 5s \le t + 3s + \frac{6(n+3)}{8} < n + 2$, since $n > 33$,

iii) $8(\pi_1 \circ T_1^{'}(s,t)) + \pi_2 \circ T_1^{'}(s,t) = 8s + t \ge 7$.

Thus $T_1'(s,t) \in \mathcal{L}^{\circ}$. Note that in the case of equality, we come back to the case (5). This completes proof of the theorem. $\square$

5 On the topology of the difference sets $K - \lambda K'$

We will use results of [MMR] in order to show the following proposition. Recall that a $R - Cantorval$ is a perfect compact subset of the real line with dense interior such that all of its bounded gaps have non-trivial intervals attached at their left extremes and have the right extremes accumulated by both intervals and gaps (see [MO]).

Proposition 2. *If n_0 is large enough, there is a residual and full measure subset $R \subset (n_0, +\infty) \times (0, +\infty)$ such that if $(n, \lambda) \in R$, then $K - \lambda K' = K(n) - \lambda K'$ is an $R - Cantorval$.*

Proof. Let us first show that for n in large as described in the previous section, and for any $\lambda > 0$, K and $\lambda K' + (n+3)$ have extremal stable intersection, i.e., there exists $\delta > 0$ such that for any pair of regular Cantor sets $(\widetilde{K}, \widetilde{K'})$ close enough to $(K, \lambda K')$, $\widetilde{K} - \widetilde{K'}$ contains an interval of size δ whose rightmost point coincides with the rightmost point $\widetilde{K} - \widetilde{K'}$. In fact, it is enough to show that, given $\lambda > 0$ there exist $\varepsilon > 0$ such that, if $n+3-\varepsilon < t < n+3$, then K and $\lambda K' + t$ have stable intersection.

In order to show this, notice that there is exactly one value of $k \in \mathbb{N}$ such that,

$$2 \leq \left(\frac{n+3}{n}\right)^k (n+3-t) < 2\left(\frac{n+3}{n}\right),$$

provided $\epsilon < 2$ (in this case, we have $0 < n+3-t < \epsilon < 2$). If $\varepsilon > 0$ is small enough, k should be large, so $(\frac{n+3}{n})^k \delta > 1$, thus there is $j \in \mathbb{N}$ such that

$$1 < \left(\frac{3}{8}\right)^j \left(\frac{n+3}{n}\right)^k \leq \frac{8}{3}.$$

Now we have:

$$T_0' T_1^k(s,t) = ((\frac{3}{8})^j (\frac{n+3}{n})^k \delta, n+3-(\frac{n+3}{n})^k (n+3-t)) =: (\widetilde{s}, \widetilde{t}) \in \mathcal{L}^{\circ},$$

indeed, $1 < \widetilde{s} \leq \frac{8}{3} < \frac{3(n+3)}{8}$ and $n < \widetilde{t} \leq n+1$, since $\frac{n+3}{n} < \frac{3}{2}$.

A pair (K, K') of two affine Cantor sets defined by increasing maps satisfies the $*$R property (see [M1] and [MMR]), if there are $\lambda > 0$, $t \in \mathbb{R}$ such that the support intervals $[a,b]$ of $\lambda K + t$ and $[c,d]$ of K are left-linked, i.e., $a < c < b < d$ and $(\lambda K + t) \bigcap K = \emptyset$. In our setting, (K, K') satisfies the $*$R property, because $\dfrac{3}{5}.[0,8] - \dfrac{19}{10}$ is left-linked with

$$[0, n+3] \text{ and } \left(\frac{3}{5}.K' - \frac{19}{10}\right) \bigcap K = \emptyset.$$

According to the arguments of proposition 1 and 2 of section 3 of [MMR], which we reproduce below, it is enough to have

$$\frac{\ln(n+3)}{\ln \frac{8}{3}} \notin \mathbb{Q} \text{ and } a - \lambda c \neq b - \lambda d, \text{ i.e., } \lambda \neq \frac{b-a}{d-c},$$

for every a, b extreme points of gaps of K and $c \neq d$ extreme points of gaps of K' in order to conclude that $K - \lambda K'$ is an R-Cantorval.

The set of (n, λ) which satisfy these conditions is the complement of a countable union of curves, so it is a residual and full-measure set.

To show the statement above, notice that a right extreme of a bounded gap of $K - \lambda K'$ is always, in a unique way, of the form $a - \lambda c$, where $a \in K, c \in K'$ are extremes of gaps, so there are intervals of the construction of K and K', say J_1 and J_2 such that a is the right extreme of J_1 and c is the left extreme of J_2, and if $\overline{K} = K \bigcap J_1$ and $\overline{K'} = K' \bigcap J_2$, then $\overline{K}$ and $\overline{K'}$ are Cantor sets similar to K and K', and, since (K, K') satisfies the $*$R property and $\frac{\ln(n+3)}{\ln \frac{8}{3}} \notin \mathbb{Q}$ therefore $\overline{K} - \lambda \overline{K'}$ does not contain any interval adjacent to $a - \lambda c$, by theorem IV.1 of [M1]. On the other hand, a left extreme of a bounded gap of $K - \lambda K'$ is of the form $a - \lambda c$, where $a \in K, c \in K'$ are extremes of gaps, say J_3 and J_4 such that a is the left extreme of J_3 and c is the left extreme of J_4. If $\widehat{K} = K \bigcap J_3$ and $\widehat{K'} = K \bigcap J_4$, then $\widehat{K}$ and $\widehat{K'}$ are Cantor sets similar to K and K' , and , by the extremal stable intersection property, $\widehat{K} - \lambda \widehat{K'}$ contains an interval attached to it rightmost point $a - \lambda c$ and this completes the proof of proposition. $\square$

Remark. The same proof shows that, for generic pairs $(\widetilde{K}, \widetilde{K'})$ of regular Cantor sets close to (K, K'), the same result holds; for $\lambda \in (0, +\infty) \backslash C$, where C is a countable set, $\widetilde{K} - \lambda \widetilde{K'}$ is a R-Cantorval.

Open Problem 1. Does there exist a non empty open set in the space of pairs of affine Cantor sets defined by two expansive maps contained in the region

$$\{(K, K') \mid \tau_R(K).\tau_L(K') < 1 \;\; \textbf{and} \;\; \tau_L(K).\tau_R(K') < 1\},$$

such that their elements have stable intersection?

We can pose the same question for homogeneous affine Cantor sets defined by two expansive maps. We find this problem particularly interesting since these are the simplest examples of regular Cantor sets, and not much is known about the arithmetic difference of two such sets in the cases where the product of their thicknesses is smaller than one but the sum of their Hausdorff dimensions is larger than one.

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